

Concealment of a Moving Object from an Observer

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Received July 22, 2013

DOI: 10.1134/S106456241306001X

1. INTRODUCTION

In the navigation of an unmanned aerial vehicle (UAV) over a ground surface elevation field in planning its trajectory of motion, one should take into account the visibility of the UAV and its concealment from an observer. It is reasonable to introduce functions characterizing the visibility and concealment of an object and establish their continuity, differentiability, and other properties.

Let G be a bodily set (the closure of an open set \mathring{G}) given in \mathbf{R}^n ; t be a moving object, $t \notin G$; and f be an observer, $f \notin \mathring{G}$. The set G hinders motion and visibility. The object t and the observer f are visible to one another if the segment $[t, f]$ does not intersect G (another definition of mutual visibility is that $[t, f] \cap \mathring{G} = \emptyset$).

The visibility function

$$r(t, f) = \min_{x \in \mathbf{R}^n} \{ \|t - x\| : [x, f] \cap G \neq \emptyset \}$$

was introduced and studied in [1, 2]. It is interpreted as the distance from t to the set of points invisible to f or, in other words, as the largest radius of a neighborhood of the object such that all the points are visible to f .

In this paper, we define two concealment functions for the object t hiding from the observer. Assume that there is a rectifiable curve $\gamma_{t, f}$ that joins the points t and f and does not intersect \mathring{G} . Let $L(\gamma_{t, f})$ denote the length of $\gamma_{t, f}$,

$$d(t, f) = \inf L(\gamma_{t, f})$$

be the infimum of the lengths of all such curves $\gamma_{t, f}$ and $\gamma(t, f)$ be the shortest curve. The function $d(t, f)$ is a metric on the set $\mathbf{R}^n \setminus \mathring{G}$. As a characteristic of the

concealment of t from the observer, we use the function (see [3])

$$c(t, f) = d(t, f) - \|t - f\|,$$

which shows as how the path from t to f in the presence of an obstacle is longer than the Euclidean distance between them.

To define another concealment characteristic, we consider the closure of the set of points in the space that are visible from t :

$$v(t, G) = \overline{\{x \in \mathbf{R}^n : [t, x] \cap G = \emptyset\}}.$$

Let $[t, f] \cap \mathring{G} \neq \emptyset$. Denote by

$$C(t, f) = d(f, v(t, G)) = \min \{d(f, x) : x \in v(t, G)\}$$

the distance from f to the set $v(t, G)$ in the metric d . The functional $C(t, f)$ also characterizes the degree of concealment of t from f : the observer has to travel over a distance at least $C(t, f)$ in order to see t .

In the case of an unfriendly observer, a moving object tries to reduce its visibility (if $[t, f] \cap G = \emptyset$) and increase its concealment (if $[t, f] \cap \mathring{G} \neq \emptyset$), while the observer's goal is always to increase visibility and reduce concealment. In this context, a problem of interest is to determine the directions in which the visibility and concealment characteristics increase (ascend) and decrease (descent); a more complicated problem is to examine the differentiability of these characteristics and to compute their directional derivatives. General information on the sets of visible and hidden points and the continuity of the functions introduced are also important for the object and the observer. Obviously, $c(t, f)$ is a continuous function of both variables, while $C(t, f)$ is a continuous function of f . The continuity of $C(t) = C(t, f)$ with respect to t is determined by the structure of the set G .

Example 1. Suppose that $G \subset \mathbf{R}^2$ consists of triangles with vertices at

$$\left(\frac{1}{k}, 1 - \frac{1}{k}\right), \left(\frac{1}{k-1}, 1\right), \left(\frac{1}{k-1}, 1 - \frac{1}{k}\right),$$

$$k = 2, 3, \dots, \quad \text{and} \quad f = (1, 0).$$

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Then $\left\{ (x, y) : x \leq 1, y \geq \max \left\{ 1 - x, \frac{1}{2} \right\} \right\}$ is the set of points invisible to the observer f . Let $t = (x, y)$ be a point such that

$$\frac{1}{k} < x < \frac{1}{k-1}, \quad y > (k-1)x$$

or

$$\frac{1}{k+1} < x < \frac{1}{k}, \quad 1-x < y < \frac{k}{k+1},$$

$$k = 2, 3, \dots$$

Let $l_{t,k}$ denote the straight line through t and the point $\left(\frac{1}{k}, 1 - \frac{1}{k}\right)$, while $\rho(f, l_{t,k})$ denote the distance from f to this line. Then $C(t) = \rho(f, l_{t,k})$. By using this fact, it is easy to show that $C(t)$ is continuous everywhere,

except for the rays $\left\{ \left(\frac{1}{k}, y\right) : y > 1 \right\}, k = 2, 3, \dots$. The

function $C(t)$ is differentiable everywhere, except for these rays and the ray $\{(0, y) : y > 1\}$.

Note that, if G is a convex subset of \mathbf{R}^n , the set-valued mapping $t \rightarrow v(t, G)$ is Hausdorff continuous in a bounded domain containing G , while the function $C(t)$ satisfies the Lipschitz condition.

In what follows, \tilde{t} and \tilde{f} denote the directions of motion of t and f , respectively.

2. DIFFERENTIAL PROPERTIES OF THE FUNCTION $c(t, f)$

Conjecture. Suppose that X is a Banach space with a differentiable norm, G is a subset of X that is the closure of an open set, the points t and f are not contained in G , and $d(t, f) < \infty$.

Then the distance $d(t, f)$ is differentiable (from one side) at t and f in any direction (\tilde{t}, \tilde{f}) .

Let $t \notin G, f \notin G, [t, f] \cap G \neq \emptyset, \gamma(t, f)$ be the shortest curve joining the points t and f , and $\gamma(t, f) \cap G = \emptyset$. Since $t \notin G$, the final segment of $\gamma(t, f)$ is a straight-line segment. Let $[t, \tilde{t}] \subset \gamma(t, f)$ be the longest segment among the indicated ones. Clearly, $\tilde{t} \in G$. In what follows, $\langle \cdot, \cdot \rangle$ denotes the scalar product of elements.

A directional differentiation formula for $c(t, f)$ can be derived only in \mathbf{R}^2 . In the general case, it is easy to establish sufficient conditions on the descent and ascent directions.

Theorem 1. Let G be a bodily set in $\mathbf{R}^2, t \notin G, f \notin G, \tilde{t}$ be a given direction, $\|\tilde{t}\| = 1$, and $t_\lambda = t + \lambda \tilde{t}$ ($\lambda \geq 0$). Then

$$\frac{d}{d\lambda} c(t_\lambda, f)_{\lambda=+0} = \frac{\langle t - \tilde{t}, \tilde{t} \rangle}{\|t - \tilde{t}\|} - \frac{\langle t - f, \tilde{t} \rangle}{\|t - f\|}.$$

In the case of an arbitrary inner-product space, \tilde{t} is a descent direction for $d(t, f)$ if $\langle t - \tilde{t}, \tilde{t} \rangle < 0$ and \tilde{t} is an

ascent direction for this function if $\left\langle \frac{t - \tilde{t}}{\|t - \tilde{t}\|}, \tilde{t} \right\rangle \geq \frac{1}{\sqrt{2}}$.

3. DESCENT DIRECTION FOR $C(t) = C(t, f)$ IN \mathbf{R}^3

Let $t, f \notin G, [t, f] \cap G = \emptyset$; and $v = v_f$ be the point of the set $v(t, G)$ nearest to f in the metric d . Note that $v \neq t$ and $(t, v) \cap G \neq \emptyset$. The following two cases are possible:

(a) There is a point w on the shortest curve $\gamma(v, f)$ such that $w \neq v$ and $[v, w] \subset \gamma(v, f)$.

(b) There is no such point. Then $v \in v(t, G) \cap G$. In this case, we assume that, at the point v , there is a one-sided half-line tangent to $\gamma(v, f)$. Denote by L the unit vector determining the tangent half-line. This vector is not parallel to the line $l = l(t, v) = \{\alpha t + (1 - \alpha)v : \alpha \in \mathbf{R}\}$. Otherwise, $\gamma(v, f)$ can be shortened by choosing a point $x \in \gamma(v, f)$ close to v and using, as the point v_f , its projection onto l .

Given a direction \tilde{t} other than that of the line l , we define a plane p containing l and parallel to the vector \tilde{t} and an ε -neighborhood $O_\varepsilon = O_\varepsilon([t, v]) \subset p$ of the segment $[t, v]$, $\varepsilon > 0$. The line l divides the set $G_\varepsilon = (O_\varepsilon \cap G) \setminus l$ into two parts. Let G_ε^+ and G_ε^- be their closures such that $\langle \tilde{t}, G_\varepsilon^- \rangle \geq 0$ and $\langle \tilde{t}, G_\varepsilon^+ \rangle \leq 0$. We use the notation $B^\pm = G_n^\pm \cap (t, v)$. The points of $[t, v]$ are arranged in increasing order from v to t . If the sets B^\pm are not empty, we define the points

$$b^+ = \max\{g \in B^+\}, \quad b_- = \min\{g \in B^-\}.$$

Theorem 2. Let $t, f \notin G, [t, f] \cap G \neq \emptyset, v = v_f$ be the point from $v(t, G)$ nearest to f , and let the vector L be defined as follows:

(a) If there is a segment $[v, w] \subset \gamma(v, f), w \neq v$, then

$$L = \frac{w - v}{\|w - v\|}.$$

(b) If there is no such segment, then it is assumed that there exists a tangent vector $L, \|L\| = 1$, to the curve $\gamma(v, f)$ at the point v .

A direction \tilde{t} satisfying the condition $\langle \tilde{t}, L \rangle < 0$ is a descent direction for the function $C(t) = C(t, f)$ in the following cases:

- (i) $B^+ = \phi$;
- (ii) $B^- = \phi$, $B^+ \neq \phi$, $b_+ > v$;
- (iii) $B^+ \neq \phi$, $B^- \neq \phi$, $b^- \leq b_+$.

Let us return to the set G (see Example 1). For all points $t \notin G$ from the triangle $\{(x, y): -\varepsilon \leq x \leq 1 + \varepsilon, 1 - x \leq y \leq 1 + \varepsilon\}$ ($\varepsilon > 0$), except for the points of the segment $\Delta = \{(x, y): x = 0, 1 < y < 1 + \varepsilon\}$, the descent and ascent directions of $C(t) = C(t, f)$ are easy to determine. For $t \in \Delta$, we have $C(t) = 1$ and any vector $\tilde{t} = \{(x, y): x < 0\}$ is a descent direction, while, in any direction $\tilde{t} = \{(x, y): x > 0\}$, the function $C(t_\lambda)$ is non-differentiable with respect to λ . Specifically, it strictly increases on the interval $\left(\frac{1}{kx}, \frac{1}{(k-1)x}\right)$, and its left limit at the point $\lambda = \frac{1}{kx}$ is strictly larger than the right limit for $k = 2, 3, \dots$. This can be shown with the help of the equality $C(t) = \rho(f, l_{t,k})$.

4. ON THE DIFFERENTIATION OF THE FUNCTION $C(t, f)$

Consider the case when, for t, f , and points close to them, the distance $d(f, v(t, G))$ is attained at a point $v_f \in v(t, G)$ such that $d(f, v_f) = \|f - v_f\|$; i.e., the segment $[f, v_f]$ is the shortest line from f to $v(t, G)$ and $[t, f] \cap \tilde{G} \neq \phi$.

Let \tilde{t} and \tilde{f} be chosen directions, $\|\tilde{t}\| = \|\tilde{f}\| = 1$, $t_\lambda = t + \lambda\tilde{t}$, $f_\lambda = f + \lambda\tilde{f}$, and $\lambda \geq 0$. For $\lambda \rightarrow 0$, there is a converging sequence of points $g_\lambda \in \text{conv}([t_\lambda, v_\lambda] \cap G)$, $g_\lambda \rightarrow g \in l(t, v_f) \cap G$, where $v_\lambda = v_{f_\lambda}$ is the point from $v(t_\lambda, G)$ nearest to f_λ and conv denotes the convex hull.

We are interested in the differentiability of the function

$$d(f_\lambda, v(t_\lambda, G))^2 = \|f_\lambda - v_\lambda\|^2$$

with respect to λ ; here,

$$v_\lambda = v^{\alpha_\lambda} = \alpha_\lambda t_\lambda + (1 - \alpha_\lambda)g_\lambda, \quad \alpha_\lambda = \alpha(t_\lambda, g_\lambda),$$

$$\alpha = \alpha(t, g) = \frac{\langle f, t - g \rangle - \langle t, g \rangle + \|g\|^2}{\|t - g\|^2}.$$

We have

$$\begin{aligned} \|f_\lambda - v_\lambda\|^2 &= \|f_\lambda\|^2 - 2\langle f_\lambda, v_\lambda \rangle + \|v_\lambda\|^2, \\ \frac{d}{d\lambda} \|f_\lambda\|_{\lambda=0}^2 &= 2\langle f, \tilde{f} \rangle. \end{aligned} \quad (1)$$

If the functionals $\|g_\lambda\|$, $\langle t, g_\lambda \rangle$, $\langle f, g_\lambda \rangle$, and $\langle \tilde{f}, g_\lambda \rangle$ are differentiable with respect to λ at the point $\lambda = 0$, then

$$\begin{aligned} \frac{d}{d\lambda} \langle f_\lambda, v_\lambda \rangle \Big|_{\lambda=0} &= \langle f, t - g \rangle \frac{d}{d\lambda} \alpha_\lambda \Big|_{\lambda=0} \\ &+ \frac{d}{d\lambda} \langle (1 - \alpha_\lambda)f, g_\lambda \rangle \Big|_{\lambda=0} + \alpha \langle f, \tilde{t} \rangle + \langle \tilde{f}, \alpha t + (1 - \alpha)g \rangle, \\ \frac{1}{2} \frac{d}{d\lambda} \|v_\lambda\|^2 &= \alpha \|t\|^2 + \left[(1 - 2\alpha) \langle f, g \rangle \right. \\ &\quad \left. - (1 - \alpha) \|g\|^2 \right] \frac{d}{d\lambda} \alpha_\lambda \Big|_{\lambda=0} \\ &+ (1 - \alpha)^2 \|g\| \frac{d}{d\lambda} \|g_\lambda\| \Big|_{\lambda=0} + \alpha^2 \langle t, \tilde{t} \rangle \\ &+ \alpha(1 - \alpha) \left[\langle \tilde{t}, g \rangle + \frac{d}{d\lambda} \langle t, g_\lambda \rangle \Big|_{\lambda=0} \right]. \end{aligned} \quad (2)$$

Theorem 3. Let \tilde{t} and \tilde{f} be given directions, $t_\lambda = t + \lambda\tilde{t}$, $f_\lambda = f + \lambda\tilde{f}$, and $[t_\lambda, f_\lambda] \cap G \neq \phi$ for small $\lambda \in [0, \lambda_*)$. If for each $\lambda \in [0, \lambda_*)$, there are points v_λ and g_λ such that $v_\lambda \in v(t_\lambda, G)$ is the nearest point to f_λ , $g_\lambda \in \text{conv}([t_\lambda, v_\lambda] \cap G)$, $[t_\lambda, v_\lambda] \cap \tilde{G} \neq \phi$, and there exists a limit of $\frac{g_\lambda - g_0}{\lambda}$ as $\lambda \rightarrow +0$, then the derivative

$$\frac{dC(t, f)}{d(\tilde{t}, \tilde{f})} = \lim_{\lambda \rightarrow +0} \frac{C(t + \lambda\tilde{t}, f + \lambda\tilde{f}) - C(t, f)}{\lambda} \quad (3)$$

in the direction (\tilde{t}, \tilde{f}) exists and is calculated using formulas (1) and (2).

To prove the theorem, it is sufficient to note that its assumptions imply the differentiability of $\langle t, g_\lambda \rangle$, $\langle f, g_\lambda \rangle$, $\|g_\lambda\|$.

Corollary. In the case of the space \mathbf{R}^n , derivative (3) exists if G is a polyhedron with a finite number of faces.

Let us derive a formula for the derivative of $C(t) = C(t, f)$ with respect to t for a convex set G . We begin with the space \mathbf{R}^2 . For $\lambda \rightarrow +0$, there are points $v \in v(t, G)$ and $v_\lambda \in v(t_\lambda, G)$ nearest to f such that $v_\lambda \rightarrow v$. The straight lines $l = l(t, v)$ and $l_\lambda = l(t_\lambda, v_\lambda)$ intersect. Let $[\bar{x}, \underline{x}] = l \cap G$, $\|t - \bar{x}\| \geq \|t - \underline{x}\|$, and n be the normal to l that is outward with respect to G . There exists a limit $\lim_{\lambda \rightarrow +0} x^\lambda = x^*$, where $x^\lambda = l \cap l_\lambda$, and it is equal to \bar{x} if $\langle n, \tilde{t} \rangle \geq 0$ and to \underline{x} if $\langle n, \tilde{t} \rangle \leq 0$. It is true that

$$\frac{dC(t)}{d\tilde{t}} = -\frac{\|v - x^*\|}{\|t - x^*\|} \langle \tilde{t}, n \rangle.$$

The case of $n \geq 3$ is analyzed by projecting the line l_λ onto the plane p passing through f and the line $l = l(t, v)$. Let ξ be the angle made by the vector \tilde{t} with the plane p , \tilde{t}_p be its projection onto p , n be the normal to l ($n \in p$)

outward with respect to $G \cap p$, and x^* be the point defined above.

Theorem 4. *Let G be a convex bodily set in \mathbf{R}^n , $[t, f] \cap \mathring{G} \neq \emptyset$, and v be the point from $v(t, G)$ nearest to f . Then the function $C(t)$ is differentiable in any direction \tilde{t} and*

$$\frac{dC(t)}{d\tilde{t}} = -\frac{\|v-x^*\|}{\|t-x^*\|} \left\langle \frac{\tilde{t}_p}{\|\tilde{t}_p\|}, n \right\rangle \cos \xi.$$

ACKNOWLEDGMENTS

This work was supported by the Presidium of the Russian Academy of Sciences under the basic research

program “Dynamical System and Control Theory,” by the Ural Branch of the Russian Academy of Sciences (project no. 12-P-1-1022), and by the Russian Foundation for Basic Research (project no. 11-01-00445).

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Translated by I. Ruzanova